Macroscopic Zeno effect and stationary flows in nonlinear waveguides with localized dissipation

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Quantum Zeno effect

- QZE — slowing down the dynamics of a quantum system subjected to frequent measurements or to a strong coupling to another quantum system.
  - theory:
  - experiment:

- More generally, ZE can be understood as the effect of changing a decay law depending on the frequency of measurements.
gas of condensed bosonic atoms

in the macroscopic dynamics the frequency of measurement can be interpreted as the strength of the induced dissipation

- V. S. Shchesnovich and V. V. Konotop, Phys. Rev. A 81, 053611 (2010)

Macroscopic ZE — the effect of the dissipation on the macroscopic characteristics of the system.
Model of a nonlinear waveguide

\[ i \psi_t = -\psi_{xx} + g|\psi|^2\psi - i\gamma(x)\psi \]

- **Localized dissipation:** \( \gamma(x) = \Gamma_0 f(x/\ell) \)
  - \( |f(x)| \to 0 \) as \( x \to \pm \infty \)
  - \( \max|f(x)| = f(0) \sim 1, \ |f_x(x)| \sim 1 \)
  - \( \Gamma_0 \) — amplitude
  - \( \ell \) — characteristic width
  - \( \int_{-\infty}^{\infty} |\gamma(x)| \, dx \propto \Gamma_0 \ell \)
Stationary flows

“hydrodynamic” form: $\Psi(t, x) = \rho(x) \exp \left[ i \int_0^x \nu(s) ds - i \mu t \right]$

- $\mu$ — chemical potential
- $n(x) = \rho^2(x)$ — density
- $\nu(x)$ — superfluid velocity
- $j(x) = \nu(x) n(x)$ — superfluid current

stationary equations:

$$\rho_{xx} + \mu \rho - g \rho^3 - j^2 \rho^{-3} = 0,$$

$$j_x + \gamma(x) \rho^2 = 0$$

boundary conditions for stationary flows:

- density is a fixed constant: $|\rho(x)| \to \rho_\infty$ as $x \to \pm \infty$ ($\rho_\infty = 1$ for all our results)
- current is some constant: $j(x) \to \mp j_\infty$ as $x \to \pm \infty$

The main question: how $j_\infty$ depends on $\Gamma_0$ and $\ell$?
\[ \rho = |\Psi| \]

\[ \int \gamma(x) dx \propto \Gamma_0 \ell \]
FIG. 1: Possible experimental scenarios to observe the MZE: nonlinear optical waveguide (upper left), a magnon waveguide (upper right), a plasmonic nanostructure (lower left), an atomic BEC in a waveguide and two reservoirs (lower right).
Example: An exact solution

\[ \rho_{xx} + \mu \rho - g \rho^3 - j^2 \rho^{-3} = 0, \]
\[ j_x + \gamma(x) \rho^2 = 0 \]

- \( \gamma(x) = 3\Gamma_0 \text{sech}^2(x/\ell) \)
- A branch of stationary flows:
  - \( \rho(x) = \tanh(x/\ell) \)
  - \( j(x) = -\Gamma_0 \ell \tanh^3(x/\ell) \)
  - additional constraint: \( \ell^2(g + \Gamma_0^2 \ell^2) = 2 \)

- \( j_\infty = \Gamma_0 \ell \)
- \( j_\infty \) grows monotonously with \( \Gamma_0 \) and \( \ell \).

- No macroscopic Zeno effect.
Another example: Dissipation with finite support

\[ \rho_{xx} + \mu \rho - g \rho^3 - j^2 \rho^{-3} = 0, \]
\[ j_x + \gamma(x) \rho^2 = 0 \]

\[ \gamma(x) = \begin{cases} 
\Gamma_0 \left(1 - x^2/\ell^2\right)^2 & \text{if } |x| < \ell, \\
0 & \text{otherwise} 
\end{cases} \]

- no exact solution; let’s start from \( \Gamma_0 = 0 \)
  - symmetric mode: \( \rho(x) = 1, \quad j(x) = 0 \)
  - antisymmetric mode: \( \rho(x) = \tanh(\sqrt{g/2}x), \quad j(x) = 0 \)

- we numerically construct “symmetric” and “antisymmetric” branches starting from \( \Gamma_0 = 0 \)
FIG. 2: (a): Density distributions $n(x)$ for symmetric flows for $g = 1$ and $\ell = 4$ and different values of $\Gamma_0$. Solid line: $\Gamma_0 = 0.01$, dashed line: $\Gamma_0 = 1$, dotted line: $\Gamma_0 = 10$. (b): Density distributions $n(x)$ for antisymmetric flows for $g = 1$ and $\ell = 1$. Solid line: $\Gamma_0 = 0.1$, dashed line: $\Gamma_0 = 1$, dotted line: $\Gamma_0 = 10$. (c) and (d): Current vs strength of the dissipation for symmetric flows (with $\ell = 4$) and for antisymmetric flows (with $\ell = 1$) obtained for $g = 0.1$ and $g = 1$; stable (unstable) flows correspond to the solid (dotted) fragments of the curves;
Dependence on the width

(e): Currents and instability increments vs width of the defect for symmetric [(s)] and antisymmetric [(a)] flows for $g = 1$ and $\Gamma_0 = 1$. In all panels $\rho_\infty = 1$. 
Two different situations:

- $\gamma(x) = 3\Gamma_0 \text{sech}^2(x/\ell)$
  - $\rho(x) = \tanh(x/\ell)$
  - $j(x) = -\Gamma_0 \ell \tanh^3(x/\ell)$
  - $j_\infty = \Gamma_0 \ell$
  - no MZE

- dissipation with finite support: different manifestations of MZE

What is the difference?

Let's look at the asymptotic behavior of the flows at $x \to \pm\infty$
Difference in asymptotic behavior

\[ \gamma(x) = 3\Gamma_0 \text{sech}^2\left(\frac{x}{\ell}\right) \]

- \[\rho(x) = \tanh\left(\frac{x}{\ell}\right)\]
- \[\rho(x) \sim 1 - 2e^{-2x/\ell} \text{ as } x \rightarrow \infty\]
- asymptotic behavior is determined by \(\ell\)

- dissipation with finite support
  - \[\rho_\infty - \rho(x) \propto e^{-\sqrt{\Lambda}x}, \text{ where } \Lambda = 2g - 4j_\infty^2\]
  - asymptotic behavior is determined by \(j_\infty\)
  - \(\Lambda > 0: j_\infty < j_\infty^{\max} = \sqrt{g/2} \) — maximal possible value of \(j_\infty\)
Making a bridge

\[ \gamma(x) = 3\Gamma_0 \text{sech}^2\left(\frac{x}{\ell}\right) \]
Making a bridge

\[ \gamma(x) = 3\Gamma_0 \text{sech}^2(\frac{x}{\ell}) \]

- \( \rho_1(x) = 1 - \rho(x), \quad \rho_1(x) \to 0 \text{ as } x \to \pm \infty \)
- \( \rho_{1,xx} - \Lambda \rho_1 = 12j_\infty \Gamma_0 \ell e^{-2x/\ell}, \quad \Lambda = 2g - 4j_\infty^2 \)
Making a bridge

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- behavior of \( \rho_1(x) \) is determined by the \( \Lambda \)-term or by r.h.s.

- In order to observe MZE we want \( \rho_1(x) \) to be governed by the \( \Lambda \)-term (as it was in the case of the dissipation with finite support)
Making a bridge

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- behavior of \( \rho_1(x) \) is determined by the \( \Lambda \)-term or by r.h.s.
- In order to observe MZE we want \( \rho_1(x) \) to be governed by the \( \Lambda \)-term (as it was in the case of the dissipation with finite support)
- two requirements:
  - \( \Lambda > 0 \) – gives the \textit{maximal} current: \( j_\infty < j_\infty^{\text{max}} = \sqrt{g/2} \)
  - \( \sqrt{\Lambda} < 2/\ell \) – gives the \textit{minimal} current: \( j_\infty > j_\infty^{\text{min}} = \sqrt{g/2 - 1/\ell^2} \)
Making a bridge

\[ \gamma(x) = 3\Gamma_0 \text{sech}^2(x/\ell) \]

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As \( \ell \) grows, \( j_\infty^{\text{min}} \) asymptotically approaches \( j_\infty^{\text{max}} \).

The range of currents allowing for MZE decreases.
Making a bridge

\[ \gamma(x) = 3\Gamma_0 \text{sech}^2(x/\ell) \]

- \( \rho_1(x) = 1 - \rho(x), \quad \rho_1(x) \to 0 \text{ as } x \to \pm\infty \)
- \( \rho_{1,xx} - \Lambda \rho_1 = 12j_\infty \Gamma_0 \ell e^{-2x/\ell}, \quad \Lambda = 2g - 4j_\infty^2 \)
- behavior of \( \rho_1(x) \) is determined by the \( \Lambda \)-term or by r.h.s.
- In order to observe MZE we want \( \rho_1(x) \) to be governed by the \( \Lambda \)-term (as it was in the case of the dissipation with finite support)
- two requirements:
  - \( \Lambda > 0 \) – gives the maximal current: \( j_\infty < j^\text{max}_\infty = \sqrt{g/2} \)
  - \( \sqrt{\Lambda} < 2/\ell \) – gives the minimal current: \( j_\infty > j^\text{min}_\infty = \sqrt{g/2 - 1/\ell^2} \)
- As \( \ell \) grows, \( j^\text{min}_\infty \) asymptotically approaches \( j^\text{max}_\infty \).
- The range of currents allowing for MZE decreases.
- Rapidly decaying dissipation is favorable for the observation of the MZE.
In summary,

- Conjecture: if the dissipation decays slower than exponentially, then there is no MZE.

- If the dissipation decays exponentially, then MZE can manifest itself but only for the stationary flows that obey a specific asymptotic behavior.

- If the dissipation decays faster than exponentially, then all the flows have a necessary asymptotics. This situation is favorable to encounter MZE.
FIG. 3: Evolution of the density $|\Psi(t, x)|^2$ starting from the initial data $\Psi(0, x) = 1$. For all the shown panels $\rho_\infty = g = \Gamma_0 = 1$. (a): The width of the defect is $\ell = 4$. The generation of the symmetric stationary flow occurs. (b) and (c): The width of the defect is $\ell = 2$ and $\ell = 6$ respectively. The symmetric flows are unstable, and therefore no stationary flow is established.
Conclusion

- Stationary flows for NLS
- MZE
- Role of the parameters of the defect
- Dynamics and possibilities of experimental observation
Appendix A: Preprint available